# CONDITIONAL AND UNCONDITIONAL LARGE GAPS BETWEEN THE ZEROS OF THE RIEMANN ZETA-FUNCTION

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Abstract. In this paper, first by employing inequalities derived from the Opial inequality due to David Boyd with best constant, we will establish new unconditional lower bounds for the gaps between the zeros of the Riemann zeta function. Second, on the hypothesis that the moments of the Hardy $Z-$function and its derivatives are correctly predicted, we establish some explicit formulae for the lower bounds of the gaps between the zeros and use them to establish some new conditional bounds. In particular it is proved that the consecutive nontrivial zeros often differ by at least 6.1392 (conditionally) times the average spacing. This value improves the value 4.71474396 that has been derived in the literature.

## 1. Introduction

The Riemann zeta function $\zeta(s)$ is defined on $\{s \in \mathbb{C} : \mathrm{Re}(s) > 1\}$ by the series

$$\zeta(s) := 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + ...,$$

which converges in the region described by the Cauchy integral test. It is of fundamental importance because it can also be represented just in terms of the primes. This representation is given by

$$\zeta(s) := \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{for} \quad \mathrm{Re}\, s > 1,$$

where the product is taken over all prime numbers. Thus its analytic properties are related to the distribution of prime numbers. Among the integers the primes appear to be scattered at random. It is known that they are infinite in number, but there is no useful formula which generate them. However, on average they obey simple laws. For example, the prime number theorem states that the number of primes which occur up to a given integer $X$, $\pi(X)$, is approximately $X/\log(X)$, the approximation getting better as $X$ increases. The actual numbers found for different $X$ will fluctuate about this value.

Riemann gave an exact formula for the counting function $\pi(X)$, in which fluctuations about the average are related to the values of $s$ for which $\zeta(s) = 0$. These are isolated points in the complex plan. In [29] the authors presented a numerical study of Riemann's formula for the oscillating part of the density of the primes and their integer powers. The formula consists of an infinite series of oscillatory terms, one for each zero of the zeta function on the critical line, and was derived by Riemann in his paper on primes, assuming the Riemann hypothesis. They also showed that high-resolution spectral lines can be generated by the truncated series at all integer powers of primes and demonstrate

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explicitly that the relative line intensities are correct. They then derived a Gaussian sum rule for Riemann's formula and used to analyze the numerical convergence of the truncated series.

Riemann conjectured that all nontrivial (non-real) zeros are distributed symmetrically with respect to the critical line $\mathrm{Re}\, s = 1/2$ and the real axis. This is the Riemann hypothesis. Riemann showed that the zeta-function satisfies a functional equation of the form

$$\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s) = \pi^{-(1-s)/2}\Gamma(\frac{1-s}{2})\zeta(1-s), \tag{1.1}$$

where $\Gamma$ is the Euler gamma function. Clearly, there are no zeros in the half-plane of convergence $\mathrm{Re}(s) > 1$, and it is also known that $\zeta(s)$ does not vanish on the line $\mathrm{Re}(s) = 1$. In the negative-half plane $\zeta(s)$ and its derivative are oscillatory and from the functional equation there exist-so called trivial (real) zeros at $s = -2m$ for any positive integer $m$ (corresponding to the poles of the appearing Gamma-factors). It is conjectured that all or at least almost all nontrivial zeros of the zeta-function are simple, see [5] and [7].

Reimann's connection between the nontrivial zeros and the primes has particularly interesting form: it bears a striking resemblance to the Gutzwiller formula, with the zeros behaving like energy levels and the primes labelling the periodic orbits of some chaotic classical system. Montgomery [24] studied the distribution of pairs of nontrivial zeros $1/2 + i\gamma$ and $1/2 + i\gamma^{'}$ and conjectured, for fixed $\alpha$, $\beta$ satisfying $0 < \alpha < \beta$, that

$$\begin{aligned} & \lim_{T\to\infty} \frac{1}{N(T)} \# \left\{ 0 < \gamma, \gamma^{'} < T : \alpha \leq \frac{\gamma^{'} - \gamma^{'}}{(2\pi/\log T)} \leq \beta \right\} \\ = & \int_{\alpha}^{\beta} \left( 1 - \left( \frac{\sin \pi x}{\pi x} \right)^2 \right) dx. \end{aligned}$$

This so-called pair correlation conjecture plays a complementary role to the Riemann hypothesis. This conjecture implies the essential simplicity hypothesis that almost all zeros of the zeta-function are simple. On the other hand the integral on the right hand side is the same as the one observed in the two point correlation of the eigenvalues which are the energy levels of the corresponding Hamiltonian that are usually not known with uncertainty. This observation is due to Dyson and it restored some hope in an old idea of Hilbert and Polya that the Riemann hypothesis follows from the existence of a self-adjoint Hermitian operator whose spectrum of eigenvalues correspond to the set of nontrivial zeros of the zeta function.

Odlyszko [28] published the results of a remarkable series of computer calculations of the zeros which showed that they were the same as those of large Hermitian matrices with randomly picked entries. These suggests that the zeros might well be the energy levels of some as yet unidentified quantum system whose classical motion is chaotic, and without symmetry under time reversal. The connections to quantum chaos and semiclassical physics are discussed in [29]. So that the distribution of zeros of the Riemann zeta-function is of fundamental importance in number theory as well as in physics.

The number $N(t)$ of the non-trivial zeros of $\zeta(s)$ with ordinate in the interval $[0,\, T]$ is asymptotically given by the Riemann-von Mangoldt formula (see [12])

$$N(T) = \frac{T}{2\pi}\log(\frac{T}{2\pi e}) + O(\log T).$$

Consequently there are infinitely many nontrivial zeros, all of them lying in the critical strip $0 < \operatorname{Re} s < 1$, and the frequency of their appearance is increasing as $T \to \infty$. Assume that $(\beta_n + i\gamma_n)$ are the zeros of $\zeta(s)$ in the upper half-plane (arranged in non-decreasing order and counted according multiplicity) and $\gamma_n \leq \gamma_{n+1}$ are consecutive ordinates of all zeros. Define

$$\lambda := \limsup_{n\to\infty} \frac{(\gamma_{n+1} - \gamma_n)}{(2\pi/\log\gamma_n)}, \text{ and } \mu := \liminf_{n\to\infty} \frac{(\gamma_{n+1} - \gamma_n)}{(2\pi/\log\gamma_n)}, \tag{1.2}$$

where $(2\pi/\log\gamma_n)$ is the average spacing between zeros. The values of $\lambda$ and $\mu$ have received a great deal of attention. In fact, important results have been obtained by some authors. It generally conjectured that

$$\mu = 0, \quad \text{and} \quad \lambda = \infty. \tag{1.3}$$

As mentioned by Montogomery [24] it would be interesting to see how numerical evidence compare with the above conjectures. Now, several results has been obtained, however the failure of Gram's low (see [19]) indicates that the asymptotic behavior is approached very slowly. Thus the numerical evidence may not be particularly illuminating. So that any numerical values of $\mu$ and $\lambda$ may be help in proving (1.3), which is one of our aims in this paper. Selberg [30] proved that $0 < \mu < 1 < \lambda$ and the average of $r_n$ is 1. Mueller [26] obtained $\lambda > 1.9$ assuming the Riemann hypothesis. Montogomery and Odlyzko [25] showed, assuming the Riemann hypothesis, that $\lambda > 1.9799$ and $\mu < 0.5179$. Conrey, Ghosh and Gonek [6] showed that, if the Riemann hypothesis is true, then $\lambda > 2.337$, and $\mu < 0.5172$. Bui, Milinovich and Ng [3] obtained $\lambda > 2.69$, and $\mu < 0.5155$ assuming the Riemann hypothesis. Conrey, Ghosh and Gonek [8] obtained a new lower bound and proved that $\lambda > 2.68$ assuming the generalized Riemann hypothesis for the zeros of the Dirichlet $L-$ functions. Ng in [27] proved that $\lambda > 3$ assuming the generalized Riemann hypothesis for the zeros of the Dirichlet $L-$functions. Bui[2] proved that $\lambda > 3.0155$ assuming the generalized Riemann hypothesis for the zeros of the Dirichlet $L-$functions. The main results in [2, 3, 8, 27] are based on the idea of Mueller [26].

Hall [13] supposed that the sequence of distinct positive zeros of the Riemann zeta-function $\zeta(\frac{1}{2} + it)$ which arranged in non-decreasing order and counted according multiplicity is given by $\{t_n\}$ and defined

$$\Lambda := \limsup_{n\to\infty} \frac{t_{n+1} - t_n}{(2\pi/\log t_n)}, \tag{1.4}$$

which is the quantity in (1.2) where only zeros $\frac{1}{2} + it_n$ on the critical line with the idea that this could be bounded from below unconditionally. Note that the Riemann hypothesis implies that the $t_n$ corresponded to the positive ordinates of non-trivial zeros of the zeta function, i.e., $N(T) \sim (T\log T)/2\pi$. The average spacing between consecutive zeros with ordinates of order $T$ is $2\pi/\log(T)$ which tends to zero as $T \to \infty$.

Hall [15] showed that $\Lambda \geq \lambda$, and the lower bound for $\Lambda$ bear direct comparison with such bounds for $\lambda$ dependent on the Riemann hypothesis, since if this were true the distinction between $\Lambda$ and $\lambda$ would be nugatory. Of course $\Lambda \geq \lambda$ and the equality holds if the Riemann hypothesis is true. So that if the Riemann hypothesis is true, we see

that any improvement of $\Lambda$ (unconditionally) will lead to the improvement of $\lambda$ and vice versa. The behavior of $\zeta(s)$ on the critical line is reflected by the Hardy $Z-$function $Z(t)$ as a function of a real variable, defined by

$$Z(t)=e^{i\theta(t)}\zeta(\frac{1}{2}+it),\ \text{where}\ \theta(t):=\pi^{-it/2}\frac{\Gamma(\frac{1}{4}+\frac{1}{2}it)}{\left|\Gamma(\frac{1}{4}+\frac{1}{2}it)\right|}. \tag{1.5}$$

Also it follows that $Z(t)$ is an infinitely often differentiable and real for real $t$ and moreover $|Z(t)|=|\zeta(1/2+it)|$. Consequently, the zeros of $Z(t)$ correspond to the zeros of the Riemann zeta-function on the critical line. In [14] Hall proved a Wirtinger-type inequality and used the moment

$$\int_0^T Z^4(t)dt=\frac{1}{2\pi^2}T\log^4(T)+O(T\log^3 T), \tag{1.6}$$

due to Ingham [21]) and the moments

$$\int_0^T (Z^{'}(t))^4dt=\frac{1}{1120\pi^2}T\log^8(T)+O(T\log^7 T), \tag{1.7}$$

$$\int_0^T Z^2(t)(Z^{'}(t))^2dt=\frac{1}{120\pi^2}T\log^6(T)+O(T\log^5 T), \tag{1.8}$$

due to Conrey [9], and obtained unconditionally that $\Lambda\geq 2.3452$.

The moments $I_k(T)$ of the Hardy $Z-$function $Z(t)$ and the moments $M_k(T)$ of its derivative are defined by

$$I_k(T):=\int_0^T |Z(t)|^{2k}dt,\ \text{and}\ M_k(T):=\int_0^T \left|Z^{'}(t)\right|^{2k}dt.$$

For positive real numbers $k$, it is believed that $I_k(T)\sim C(k)\ T\left(\log T\right)^{k^2}$ and $M_k(T)\sim L(k)T\left(\log T\right)^{k^2+2k}$ for positive constants $C_k$ and $L_k$ will be defined later. Keating and Snaith [22] based on considerations from random matrix theory conjectured that

$$I_k(T)\sim a(k)b(k)T\left(\log T\right)^{k^2}, \tag{1.9}$$

where $a(k)$ is a product over the primes which is defined by

$$a(k):=\prod_p(\left(1-\frac{1}{p^2}\right)\sum_{m=0}^{\infty}\left(\frac{\Gamma(m+k)}{m!\Gamma(k)}\right)^2 p^{-m},$$

and

$$b(k):=\prod_{j=0}^{k-1}\frac{j!}{(j+k)!}. \tag{1.10}$$

Using the relation (1.10) one can obtain the value of $b(k)$ for any real positive number $k$. Conrey, Rubinstein and Snaith [10] conjectured that

$$M_k(T)\sim a(k)c(k)T\left(\log T\right)^{k^2+2k}, \tag{1.11}$$

where

$$c(k):=(-1)^{\frac{k(k+1)}{2}}\sum_{m\in P_O^{k+1}(2k)}^{k}\begin{pmatrix}2k\\ m\end{pmatrix}\left(\frac{-1}{2}\right)^{m_0}\left(\prod_{i=1}^{k}\tfrac{1}{(2k-i+m_i)!}\right)M_{i,j},$$

where

$$M_{i,j} := \left( \prod_{1\le i,j\le k}^{k} (m_j - m_i + i - j) \right), \tag{1.12}$$

and $P_O^{k+1}(2k)$ denotes the set of partitions $m = (m_0, ..., m_k)$ of $2k$ into nonnegative parts. In this paper, we will determine the values of $b(k)/c(k)$ for $k = 1, 2, ..., 15$ that we will use derive the new conditional lower bounds for $\Lambda$.

Hall in [15, 18] used the moments of mixed powers of the form

$$\int_0^T Z^{2k-2h}(t)(Z^{'}(t))^{2h} dt \sim C(h,k)T\,(\log T)^{k^2+2h}, \tag{1.13}$$

where $Z(t)$ is the Hardy $Z-$function, and $k \in N$, $0 \le h \le k$ and a complicated variation problem together with a new Wirtinger-type inequality designed exclusively for this problem and obtained some conditional lower bounds of $\Lambda$. The moments in (1.13) has been predicted by Random Matrix Theory (RMT) by Hughes [20] who stated an interesting conjecture on the moments of the zeta function and its derivatives at its zeros subject to the truth of Riemann's hypothesis when the zeros are simple. This conjecture includes for fixed $k > -3/2$ the asymptotes formula of the moments of the higher order of the Riemann zeta function and its derivative. We suppose further that if $k$ is a fixed positive integer and $h \in [0,\, k]$ is an integer then the formula

$$\int_0^T Z^{2k-2h}(t)(Z^{'}(t))^{2h} dt \sim a(k)b(h,k)T\,(\log T)^{k^2+2h}, \tag{1.14}$$

holds. Note that this was predicted by Keating and Snaith [22] in the case when $h = 0$, with wider range $\mathrm{Re}(k) > -1/2$ and by Hughes [20] in the range $\min(h, k-h) > -1/2$, $a(k)$ is a product over the primes and $b(h,k)$ is rational: indeed for integral $h$, it is obtained that

$$b(h,k) = b(k)\left(\frac{(2h)!}{8^h h!}\right) H(h,k), \tag{1.15}$$

where $H(h,k)$ is an explicit rational function of $k$ for each fixed $h$. The functions $H(h,k)$ as introduced by Hughes [20] are given in the following table where $K = 2k$:

| |
|---|
| $H(0,k) = 1,\ H(1,k) = \frac{1}{K^2-1},\ H(2,k) = \frac{1}{(K^2-1)(K^2-9)}$ |
| $H(3,k) = \frac{1}{(K^2-1)^2(K^2-25)},\ H(4,k) = \frac{K^2-33}{(K^2-1)^2(K^2-9)(K^2-25)(K^2-49)}$ |
| $H(5,k) = \frac{K^4-90K^2+1497}{(K^2-1)^2(K^2-9)^2(K^2-25)(K^2-49)(K^2-81)},$ |
| $H(6,k) = \frac{K^6-171K^4+6867K^2-27177}{(K^2-1)^3(K^2-9)^2(K^2-25)(K^2-49)(K^2-81)(K^2-121)},$ |
| $H(7,k) = \frac{K^8-316K^6+30702K^4-982572K^2+6973305}{(K^2-1)^3(K^2-9)^2(K^2-25)^2(K^2-49)(K^2-81)(K^2-121)(K^2-169)},$ |

*Table 1. The values of $H(h,k)$, where $K = 2k$.*

This sequence continuous, and it is believed that both the nominator and denominator are polynomials in $k^2$, moreover that the denominator is actually (see [11])

$$\prod_{a\ odd>0} \left\{ (K^2 - a^2)^{\alpha(a,h)} : \alpha(a,h) = \frac{4h}{a + \sqrt{a^2 + 8h}} \right\}. \tag{1.16}$$

Using the equation (1.15) and the definitions of the functions $H(h,k)$, we can obtain the values of $b(0,k)/b(k,k)$ for $k = 1,2,...,7$. As indicated in [18] Hughes [20] conjectured the first four functions and then writes that numerical experiment suggests the next three. Hall [18] shown that in the case when $h = 3$, $(H(3,k))$ requires adjustment to fit with (1.16) in that extra factor $K^2 - 9$ should be introduced in both the nominator and denominator. To use (1.14) Hall [15] proved a new generalized Wirtinger-type inequality of the form

$$\int_0^\pi H\left(y^{'}(t)/y(t)\right) y^{2k}(t)dt \geq (2k-1)L\int_0^\pi y^{2k}(t)dt, \tag{1.17}$$

where $y(t) \in C^2[0,\,\pi]$, $y(0) = y(\pi) = 0$, $L = L(k,H)$ is determined from the solution of the equation

$$\int_0^\infty \frac{G^{'}(u)}{G(u)+(2k-1)L}\frac{du}{u} = k\pi, \text{ for } k \in \mathbb{N},$$

where $G(u) := uH^{'}(u) - H(u)$, $H(u)$ be an even function, increasing, strictly convex on $\mathbb{R}^+$ and satisfies $H(0) = H^{'}(0) = 0$,and $uH^{''}(u) \to 0$ as $u \to 0$. The inequality (1.17) is proved by using the calculus of variation which depends on the minimization of the integral on the left hand side subject to the constrains $y(0) = 0$ and $\int_0^\pi y^{2k}(t)dt = 1$. Assuming that (1.14) is correctly predicted, Hall employed the inequality (1.17) when

$$H(u) := \sum_{h=1}^{k} \frac{2k-1}{2h-1}\left(\begin{array}{c} h \\ k \end{array}\right)\upsilon_h u^{2h}, \quad \upsilon_h \geq 0, \quad \upsilon_k = 1,$$

and obtained an explicit formula for $\Lambda(k)$ which is given by

$$\Lambda^2 \geq X,$$

where $X$ is the real positive root of the equation

$$\sum_{h=1}^{k} \frac{2k-1}{2h-1}\left(\begin{array}{c} k \\ h \end{array}\right) R(h,k)\upsilon_h X^h - (2k-1)\lambda(\upsilon_1,\upsilon_1,...,\upsilon_{k-1})R(0,k) = 0,$$

and

$$R(h,k) := 4^{k-h}\frac{b(h,k)}{b(k,k)}.$$

He then derived a new value of $\Lambda$ (when $k = 3$) which is given by

$$\Lambda \geq \sqrt{7533/901} = 2.8915. \tag{1.18}$$

The main challenge in [15] was to maximize $X = \kappa^2$ (which is not an easy task) where $X$ satisfies the equation

$$27X^3 + 385\mu X^2 + 10395\vartheta X - 121275L = 0,$$

and $L$ obtained form the equation

$$\int_{-\infty}^{\infty} \frac{x^4 + 2\mu x^2 + \upsilon}{x^6 + 3\mu x^4 + 3\upsilon x^2 + L}dx = \pi.$$

Hall [16] simplified the calculations in [15] and converted the problem into one of the classical theory of equations involving Jacobi-Schur functions and proved that $\Lambda(4) \geq 3.392272$, $\Lambda(5) \geq 3.858851$, and $\Lambda(6) \geq 4.2981467$. Hall [18] developed the theory set used in [16] and proved that $\Lambda(7) \geq 4.215007$ assuming that (1.14) is correctly predicted. The improvement of this value as obtained in [18] is given by $\Lambda(7) \geq 4.71474396$ assuming that (1.14) is correctly predicted. The question now is: If it is possible to employ new

inequalities with best constants to find new explicit formulae for the gaps and use them to find new series of the lower bounds?

The paper gives an affirmative answer to this question. In fact, we will derive new unconditional and conditional lower bounds for $\Lambda$. The main results will be proved by employing two inequalities derived from the Opial inequality with a best constant due to David Boyd [4] who applied a variational technique to reduce the determination of the best constant to a nonlinear eigenvalue problem for an integral operator.

The main results will be proved in the next section which is organized as follows: First, we derive new unconditional lower bounds for $\Lambda$. Second on the hypothesis that the moments of the Hardy $Z-$function and its derivatives are correctly predicted, we establish new explicit formulae of the gaps between the zeros and establish some lower bounds for $\Lambda$. In particular, we will prove that $\Lambda \geq 6.1392$ which improves the value $\Lambda \geq 4.71474396$.

## 2. Main Results

Before we state and prove the main results, we derive some inequalities from the Opial inequality due to David Boyd [4] that we will use in this section. The Opial inequality due to David Boyd [4] is presented in the following theorem.

**Theorem A.** *If $y \in C^1[a, b]$ with $y(a) = 0$ (or $y(b) = 0$), then*

$$\int_a^b |y(t)|^p \left|y^{'}(t)\right|^q dt \leq K(p,q,r)(b-a)^{r-q} \left(\int_a^b \left|y^{'}(t)\right|^r dt\right)^{\frac{p+q}{r}}, \tag{2.1}$$

*where $p > 0$, $r > 1$, $0 \leq q < r$,*

$$\begin{aligned} K(p,q,r) \quad &: \quad = \frac{(r-q)\,p^p}{(r-1)(p+q)} \beta^{p+q-r} \left(I(p,q,r)\right)^{-p}, \\ \beta \quad &: \quad = \left\{\frac{p(r-1)+(r-q)}{(r-1)(p+q)}\right\}^{\frac{1}{r}}, \end{aligned}$$

*and*

$$I(p,q,r) := \int_0^1 \left\{1 + \frac{r(q-1)}{r-q} t\right\}^{-(p+q+rp)/rp} [1+(q-1)t] t^{1/p-1} dt.$$

First, we will derive a new inequality from the Opial inequality (2.1) of the from (1.17) which allows us to use the moments (1.7) and (1.8) to derive the new unconditional lower bound of $\Lambda$. For a special case of Theorem A, when $r = p+q$, we have

$$\int_a^b |y(t)|^p \left|y^{'}(t)\right|^q dt \leq K(p,q,p+q)(b-a)^p \left(\int_a^b \left|y^{'}(t)\right|^{p+q} dt\right), \tag{2.2}$$

where

$$K(p,q,p+q) := \frac{q(p+q)^{p-1}}{(pL(p,q)+q)^p}, \quad q \neq 0, \tag{2.3}$$

and

$$L(p,q) := \int_0^1 \left(\frac{1}{1-\lambda s^p}\right) ds, \quad \text{where } \lambda = \frac{(p+q)(q-1)}{(p+q-1)q}. \tag{2.4}$$

The inequality (2.2) has immediate application to the case where $y(a) = y(b) = 0$. Choose $c = (a+b)/2$ and apply (2.12) to $[a,c]$ and $[c,b]$ and then add to obtain

$$\begin{aligned}
&\int_a^b |y(t)|^p \left|y^{'}(t)\right|^q dt \\
\leq\ & K(p,q,p+q)(\frac{b-a}{2})^p \left(\int_a^c \left|y^{'}(t)\right|^{p+q} dt + \int_c^b \left|y^{'}(t)\right|^{p+q} dt\right) \\
\leq\ & K(p,q,p+q)(\frac{b-a}{2})^p \left(\int_a^b \left|y^{'}(t)\right|^{p+q} dt\right).
\end{aligned}$$

So that if $y(0) = y(\pi) = 0$, we have

$$\int_0^\pi |y(t)|^p \left|y^{'}(t)\right|^q dt \leq K(p,q,p+q)(\frac{\pi}{2})^p \left(\int_0^\pi \left|y^{'}(t)\right|^{p+q} dt\right). \tag{2.5}$$

If we choose $p = 2$ and $q = 2$, we get that

$$\int_0^\pi y^2(t)(y^{'}(t))^2 dt \leq K(2,2,4)(\frac{\pi}{2})^2 \left(\int_0^\pi \left(y^{'}(t)\right)^4 dt\right). \tag{2.6}$$

Using the definition of $K$, we see that $K(2,2,4) = 0.346\,13$, where we used the value of

$$L(2,2) = \int_0^1 \left(\frac{1}{1-\frac{2}{3}s^2}\right) ds = 1.4038.$$

So that the inequality (2.6) becomes

$$\left(\int_0^\pi \left(y^{'}(t)\right)^4 dt\right) \geq \frac{4}{(0.346\,13)\,\pi^2} \int_0^\pi y^2(t)(y^{'}(t))^2 dt. \tag{2.7}$$

By a suitable linear transformation, we deduce that if $y \in C^1[a,b]$ with $y(a) = 0 = y(b)$, then we have

$$\int_a^b \left(\frac{b-a}{\pi}\right)^2 \left(y^{'}(t)\right)^4 dt \geq \frac{4}{(0.346\,13)\,\pi^2} \int_a^b y^2(t)(y^{'}(t))^2 dt. \tag{2.8}$$

In the following, assuming the Riemann hypothesis, we will apply the inequality (2.8) and using the moments (1.7) and (1.8) to find a new unconditional value of $\Lambda$. One can see that the value that we will establish does not improve the obtained values, but the technique is a simple one and depends only on the application of an inequality derived from the well-known Opial inequality. Note that, as mentioned by Hall, when the Riemann hypothesis is true we have $\lambda = \Lambda$.

**Theorem 2.1**. *Let $\varepsilon(T) \to 0$ in such a way that $\varepsilon(T)\log T \to \infty$. Then for sufficiently large $T$, there exists an interval contained in $[T, (1+\varepsilon(T))T]$ which is free of zeros of $Z(t)$ and having length at least*

$$\sqrt{\frac{112}{12\,(0.34613)\,\pi^2}} \left\{1 + O\left(\frac{1}{\varepsilon(T)\log T}\right)\right\} \frac{2\pi}{\log T}.$$

*Thus*

$$\Lambda \geq \frac{1}{\pi}\sqrt{\frac{112}{12\,(0.34613)}} = 1.6529. \tag{2.9}$$

**Proof.** We follow the arguments in [14] to prove our theorem. Suppose that $t_l$ is the first zero of $Z(t)$ not less than $T$ and $t_m$ the last zero not greater than $(1+\varepsilon)T$ where $\varepsilon(T) \to 0$ in such a way that $\varepsilon(T)\log T \to \infty$. Suppose further that for $l \leq n < m$, we have

$$L_n = t_{n+1} - t_n \leq \frac{2\pi\kappa}{\log T}. \tag{2.10}$$

Applying the inequality (2.8) with $y(t) = Z(t)$, we have

$$\int_{t_n}^{t_{n+1}} \left[ \left(\frac{L_n}{\pi}\right)^4 \left(Z^{'}(t)\right)^4 - \frac{4}{(0.346\,13)\,\pi^2} Z^2(t)(Z^{'}(t))^2 \right] dt \geq 0.$$

Since the inequality remains true if we replace $L_n/\pi$ by $2\kappa/\log T$, we have

$$\int_{t_n}^{t_{n+1}} \left[ \left(\frac{2\kappa}{\log T}\right)^4 \left(Z^{'}(t)\right)^4 - \frac{4}{(0.346\,13)\,\pi^2} Z^2(t)(Z^{'}(t))^2 \right] dt \geq 0. \tag{2.11}$$

Summing (2.11) over $n$, using (1.7) and (1.8), we obtain

$$\begin{aligned} & \frac{1}{1120\pi^2}\left(\frac{2\kappa}{\log T}\right)^2 T\log^8(T) + O(T\log^7) \\ & -\frac{4}{(0.346\,13)\,\pi^2}\frac{1}{120\pi^2} T\log^6(T) + O(T\log^5 T) \\ = \; & \frac{(2\kappa)^2}{1120\pi^2} T\log^6(T) + O(T\log^7 T) \\ & -\frac{4}{(0.346\,13)\,\pi^2}\frac{1}{120\pi^2}(T\log^6 T) + O(T\log^5 T). \end{aligned}$$

Follows the proof of Theorem 1 in [14], we obtain

$$\kappa^2 \geq \frac{112}{12\,(0.346\,13)\,\pi^2} + O(1/\varepsilon(T)\log T).$$

Then, we have (noting $(\varepsilon(T)\log T \to \infty$ as $T \to \infty)$ that

$$\Lambda \geq \sqrt{\frac{112}{12\,(0.34613)\,\pi^2}} = 1.652\,9,$$

which the desired value (2.9). The proof is complete.

Next in the following, we will derive an inequality from the Opial inequality (2.1) which allows use to use the moments (1.6) and (1.7) ((1.14)) to derive a new unconditional (conditional) lower bound for $\Lambda$. As a special case of ((2.1) if $q = 0$, and $p = r = 2k$, then the inequality (2.1) reduces to

$$\int_a^b |y(t)|^{2k}\, dt \leq A^{2k}(k)\,(b-a)^{2k} \int_a^b \left|y^{'}(t)\right|^{2k} dt, \tag{2.12}$$

where

$$A(k) := \left(\frac{2(k)}{2(k)-1}\right)^{\frac{1}{2k}} (2(k))^{\frac{2(k)-1}{2(k)}} \left(\Gamma(\frac{1}{2(k)})\Gamma(\frac{2(k)-1}{2(k)})\right)^{-1}, \tag{2.13}$$

and $\Gamma(u)$ is the Euler gamma function. The inequality (2.12) has immediate application to the case where $y(a) = y(b) = 0$. Choose $c = (a+b)/2$ and apply (2.12) to $[a,\, c]$ and $[c,\, b]$ and then add to obtain

$$\begin{aligned}\int_a^b |y(t)|^{2k}\, dt &\leq A^{2k}(k)\left(\frac{b-a}{2}\right)^{2k}\left\{\left(\int_a^c \left|y^{'}(t)\right|^{2k} dt\right)+\left(\int_c^b \left|y^{'}(t)\right|^{2k} dt\right)\right\} \\ &\leq A^{2k}(k)\left(\frac{b-a}{2}\right)^{2k}\int_a^b \left|y^{'}(t)\right|^{2k} dt. \qquad (2.14)\end{aligned}$$

From this, we have

$$\int_0^{\pi} |y(t)|^{2k}\, dt \leq A^{2k}(k)\left(\frac{\pi}{2}\right)^{2k}\int_0^{\pi} \left|y^{'}(t)\right|^{2k} dt.$$

with $y(0) = 0 = y(\pi)$. From this inequality, we deduce that if $y \in C^1[a,b]$ with $y(a) = 0 = y(b)$, then we have

$$\int_a^b \left(\frac{b-a}{\pi}\right)^{2k}\left(y^{'}(t)\right)^{2k} dt \geq \left(\frac{2}{\pi}\right)^{2k}\frac{1}{A^{2k}(k)}\int_a^b (y(t))^{2k} dt. \qquad (2.15)$$

Using the formula (2.13), we have the following values of $A(k)$ for $k = 1, 2, ....15$.

| | | | | |
|---|---|---|---|---|
| $A(1)$ | $A(2)$ | $A(3)$ | $A(4)$ | $A(5)$ |
| 0.63662 | 0.68409 | 0.73026 | 0.76409 | 0.7896 |
| $A(6)$ | $A(7)$ | $A(8)$ | $A(9)$ | $A(10)$ |
| 0.80955 | 0.825 62 | 0.83888 | 0.85003 | 0.85956 |
| $A(11)$ | $A(12)$ | $A(13)$ | $A(14)$ | $A(15)$ |
| 0.8678 | 0.87502 | 0.88141 | 0.88709 | 0.89219 |

Table 2. The values of $A(k)$ for $k = 1, 2, ..., 15$.

In the following theorem, assuming the Riemann hypothesis, we apply the inequality (2.15) when $k = 2$ and using the moments (1.6) and (1.7) to derive an unconditional value for $\Lambda$.

**Theorem 2.2**. *Let $\varepsilon(T) \to 0$ in such a way that $\varepsilon(T)\log T \to \infty$. Then for sufficiently large $T$, there exists an interval contained in $[T, (1+\varepsilon(T))T]$ which is free of zeros of $Z(t)$ and having length at least*

$$\frac{1}{\pi\,(0.68409)}\sqrt[4]{560}\left\{1+O\left(\frac{1}{\varepsilon(T)\log T}\right)\right\}\frac{2\pi}{\log T}.$$

*Thus*

$$\Lambda \geq \frac{1}{\pi\,(0.68409)}\sqrt[4]{560} = 2.2635. \qquad (2.16)$$

**Proof.** As in the proof of Theorem 2.1, we follow the arguments in [14] to prove our theorem. Suppose that $t_l$ is the first zero of $Z(t)$ not less than $T$ and $t_m$ the last zero not greater than $(1+\varepsilon)T$ where $\varepsilon(T) \to 0$ in such a way that $\varepsilon(T)\log T \to \infty$. Suppose further that for $l \leq n < m$, we have

$$L_n = t_{n+1} - t_n \leq \frac{2\pi\kappa}{\log T}. \qquad (2.17)$$

Applying the inequality (2.15) with $y(t) = Z(t)$ and $k = 2$, we have

$$\int_{t_n}^{t_{n+1}} \left[ \left(\frac{L_n}{\pi}\right)^4 \left(Z^{'}(t)\right)^4 - \left(\frac{2}{\pi A(2)}\right)^4 (Z(t))^4 \right] dt \geq 0.$$

Since the inequality remains true if we replace $L_n/\pi$ by $2\kappa/\log T$, we have

$$\int_{t_n}^{t_{n+1}} \left[ \left(\frac{2\kappa}{\log T}\right)^4 \left(Z^{'}(t)\right)^4 - \left(\frac{2}{\pi A(2)}\right)^4 (Z(t))^4 \right] dt \geq 0. \tag{2.18}$$

Summing (2.18) over $n$, using (1.6) and (1.7), we obtain

$$\begin{aligned} & \frac{1}{1120\pi^2} \left(\frac{2\kappa}{\log T}\right)^4 T \log^8(T) + O(T \log^7) \\ & - \left(\frac{2}{\pi A(2)}\right)^4 \frac{1}{2\pi^2} (T \log^4 T) + O(T \log^3 T) \\ = \; & \frac{(2\kappa)^4}{1120\pi^2} T \log^4(T) + O(T \log^7 T) \\ & - \left(\frac{2}{\pi A(2)}\right)^4 \frac{1}{2\pi^2} (T \log^4 T) + O(T \log^3 T). \end{aligned}$$

Follows the proof of Theorem 1 in [14], we obtain

$$\kappa^4 \geq \frac{1}{2^4} \left(\frac{2}{\pi A(2)}\right)^4 \frac{1120\pi^2}{2\pi^2} + O(1/\varepsilon(T) \log T).$$

Now, using the value of $A(2)$ from Table 2, we have (noting $(\varepsilon(T) \log T \to \infty$ as $T \to \infty)$ that

$$\Lambda \geq \frac{1}{\pi} \frac{1}{(0.68409)} \sqrt[4]{\frac{1120}{2}} = 2.2635,$$

which the desired value (2.16). The proof is complete.

In the following, we will establish some explicit formulae for the gaps between the zeros of the Riemann zeta function and use them to find new conditional series of lower bounds. First, we will apply the inequality (2.5). As usual, we assume that the Riemann hypothesis is true. As a special case of (2.5), if $y(0) = y(\pi) = 0$, $p = 2k - 2h$ and $q = 2h$, we have

$$\int_0^{\pi} |y(t)|^{2k-2h} \left|y^{'}(t)\right|^{2h} dt \leq K(h,k)(\frac{\pi}{2})^{2k-2h} \left(\int_0^{\pi} \left|y^{'}(t)\right|^{2k} dt\right), \tag{2.19}$$

where

$$K(h,k) = \frac{hk^{2k-1}}{((k-h)L(k,h) + h)^{2k}}, \quad h, \ k \neq 0, \tag{2.20}$$

and

$$L(h,k) = \int_0^1 \left(\frac{1}{1 - \lambda s^{2k-2h}}\right) ds, \quad \lambda = \frac{k(2h-1)}{h(2k-1)}. \tag{2.21}$$

**Theorem 2.3.** *On the hypothesis that the Riemann hypothesis is true and (1.14) is correctly predicted, we have*

$$\Lambda \geq \Lambda^*(h,k) := \frac{1}{\pi} \left(\frac{1}{K(h,k)} \frac{b(h,k)}{b(k,k)}\right)^{\frac{1}{2k-2h}}, \ h \neq k \neq 0 \tag{2.22}$$

*where*

$$b(h,k) := \frac{2h!}{8^h h!} H(h,k) \left( \prod_{j=0}^{k-1} \frac{j!}{(j+k)!} \right). \tag{2.23}$$

**Proof.** As in the proof of Theorem 2.2 by applying the inequality (2.19) with $y = Z(t)$, we have

$$\int_{t_n}^{t_{n+1}} \left[ \left( \frac{L_n}{\pi} \right)^{2k} \left( Z'(t) \right)^{2k} - \frac{2^{2k-2h}}{K(h,k)\pi^{2k-2h}} \left( \frac{L_n}{\pi} \right)^{2h} |Z(t)|^{2k-2h} \left| Z'(t) \right|^{2h} \right] dt \geq 0.$$

Since the inequality remains true if we replace $L_n/\pi$ by $2\kappa/\log T$, we have

$$\int_{t_n}^{t_{n+1}} \left( \frac{2\kappa}{\log T} \right)^{2k} \left| Z'(t) \right|^{2k}$$
$$- \int_{t_n}^{t_{n+1}} \frac{2^{2k-2h}}{K(h,k)\pi^{2k-2h}} \left( \frac{2\kappa}{\log T} \right)^{2h} |Z(t)|^{2k-2h} \left| Z'(t) \right|^{2h} dt \geq 0.$$

Summing (2.25) over $n$ and using (1.14), we obtain

$$\left( \frac{2\kappa}{\log T} \right)^{2k} a(k)b(k,k)T (\log T)^{k^2+2k}$$
$$- \frac{2^{2k-2h}}{K(h,k)\pi^{2k-2h}} \left( \frac{2\kappa}{\log T} \right)^{2h} a(k)b(h,k)T (\log T)^{k^2+2h} dt \geq 0.$$

This implies that

$$\begin{aligned} & T (\log T)^{k^2} \left\{ (2\kappa)^{2k} a(k)b(k,k) - \frac{2^{2k-2h}}{K(h,k)\pi^{2k-2h}} (2\kappa)^{2h} a(k)b(h,k) \right\} \\ \geq \quad & o(T (\log T)^{k^2}), \end{aligned}$$

whence

$$\kappa^{2k-2h} \geq \frac{1}{\pi^{2k-2h} K(h,k)} \frac{b(h,k)}{b(k,k)} + o(1), \ \text{(as } T \to \infty).$$

This implies that

$$\Lambda^{2k-2h}(k) \geq \frac{1}{\pi^{2k-2h} K(h,k)} \frac{b(h,k)}{b(k,k)}, \ \ h \neq k \neq 0.$$

which is the desired inequality and completes the proof.

To apply the inequality (2.22), we will need the following values of $b(1,k)$ and $b(k,k)$ which are determined from (2.23) where $H(h,k)$ are defined as in Table 1:

$$\begin{aligned} b(1,2) &= \tfrac{1}{720}, \ b(2,2) = \tfrac{1}{6720}, \ b(1,3) = \tfrac{1}{1209\,600}, \ b(3,3) = \tfrac{1}{496742400}, \\ b(1,4) &= \tfrac{1}{219469\,824\,000}, \quad b(4,4) = \tfrac{31}{271159356948480000}, \\ b(1,5) &= \tfrac{1}{8760533070643200\,000}, \ b(5,5) = \tfrac{227}{12854317559387145633792000000}, \\ b(1,6) &= \tfrac{1}{127288\,050\,516627\,176\,816640\,000\,000}, \\ b(6,6) &= \tfrac{133933}{2551645909444410418740124199996620800000000}, \\ b(1,7) &= \tfrac{1}{99870792607969510161119437833011200000000000}, \\ b(7,7) &= \tfrac{2006509}{895370835179\,281010\,419215815294340559070476369920000000000000}. \end{aligned}$$

Table 3: The values of $b(1,k)$ and $b(k,k)$ for $k=2,...,7$.

Also, we need the following values of $K(1,k)$ which are determined from the formula (2.20) by using (2.21) for $k=2,3,...,7$ :

$$\begin{aligned} K(1,2) &= 0.239\,61,\ K(1,3)=0.161\,87,\ K(1,4)=0.122\,27, \\ K(1,5) &= 9.823\,8\times 10^{-2},\ K(1,6)=8.212\,8\times 10^{-2},\ K(1,7)=0.07055. \end{aligned}$$

Now, we are ready to derive a series of the lower bounds of $\Lambda(k)$ for $k=2,3,...,7$. These lower bounds are determined by using the formula (2.22) and presented in the following table:

| $\Lambda(2)$ | $\Lambda(3)$ | $\Lambda(4)$ | $\Lambda(5)$ | $\Lambda(6)$ | $\Lambda(7)$ |
|---|---|---|---|---|---|
| 1.9866 | 2.2591 | 2.6407 | 3.0208 | 3.3800 | 3.7124 |

Table 3: The lower bounds for $\Lambda(k)$ for $k=2,3,...,7$ by using the formula (2.22).

In the following, we will apply the inequality (2.15) to establish a new explicit formula for $\Lambda(k)$. As usual, we assume that the Riemann hypothesis is true and the moments in (1.9) and (1.11) are correctly predicted.

**Theorem 2.4**. *Assuming the Riemann hypothesis and the moments in (1.9) and (1.11) are correctly predicted, we have*

$$\Lambda(k)\geq \frac{1}{\pi A(k)}\left(\frac{b_k}{c_k}\right)^{\frac{1}{2k}},\quad \text{for } k\geq 3, \tag{2.24}$$

*where $A(k)$ is defined as in (2.13).*

**Proof.** As in the proof of Theorem 2.2 by applying the inequality (2.15) with $y=Z(t)$, we have

$$\int_{t_n}^{t_{n+1}}\left[\left(\frac{L_n}{\pi}\right)^{2k}\left(Z^{'}(t)\right)^{2k}-\left(\frac{2}{\pi A(k)}\right)^{2k}(Z(t))^{2k}\right]dt\geq 0.$$

Since the inequality remains true if we replace $L_n/\pi$ by $2\kappa/\log T$, we have

$$\int_{t_n}^{t_{n+1}}\left[\left(\frac{2\kappa}{\log T}\right)^{2k}\left(Z^{'}(t)\right)^{2k}-\left(\frac{2}{\pi A(k)}\right)^{2k}(Z(t))^{2k}\right]dt\geq 0. \tag{2.25}$$

Summing (2.25) over $n$, using (1.9) and (1.11), we obtain

$$\begin{aligned} & a_kc_k\left(\frac{2\kappa}{\log T}\right)^{2k}T\left(\log T\right)^{k^2+2k}-a_kb_k\left(\frac{2}{\pi A(k)}\right)^{2k}T\left(\log T\right)^{k^2} \\ = & \left(a_kc_k\kappa^{2k}(2^{2k})-a_kb_k\left(\frac{2}{\pi A(k)}\right)^{2k}\right)T\left(\log T\right)^{k^2}\geq O(T\log^{k^2}T), \end{aligned}$$

whence

$$\kappa^{2k}\geq \frac{a_kb_k}{2^{2k}a_kc_k}\frac{1}{A^{2k}(k)}=\frac{b_k}{2^{2k}c_k}\left(\frac{2}{\pi A(k)}\right)^{2k},\ (\text{as } T\to\infty).$$

This implies that

$$\Lambda^{2k}(k)\geq \frac{b_k}{2^{2k}c_k}\left(\frac{2}{\pi A(k)}\right)^{2k},$$

and then we obtain the desired inequality (2.24). The proof is complete.

Conrey, Rubinstein and Snaith [10] gave some explicit values of the parameter $c(k)$ for $k = 1, 2, ..., 15$. Using the values of $c(k)$ due to Conrey, Rubinstein and Snaith [10] and the relation (1.10), we have the following values of $b(k)/c(k)$ for $k = 1, 2, ..., 15$ that will be used in this paper:

$$
\begin{aligned}
\frac{b_1}{c_1} &= 2^2 \cdot 3, \quad \frac{b_2}{c_2} = \frac{2^6 \cdot 3 \cdot 5 \cdot 7}{2^2 3}, \quad \frac{b_3}{c_3} = \frac{2^{12} \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11}{2^6 3^3 5}, \quad \frac{b_4}{c_4} = \frac{2^{20} \cdot 3^{10} \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13}{31 \cdot 2^{12} 3^5 5^3 7}, \\
\frac{b_5}{c_5} &= \frac{2^{30} \cdot 3^{12} \cdot 5^6 \cdot 7^4 \cdot 11 \cdot 13^2 \cdot 17 \cdot 19}{227 \cdot 2^{20} 3^9 5^5 7^3}, \quad \frac{b_6}{c_6} = \frac{2^{42} \cdot 3^{19} \cdot 5^9 \cdot 7^6 \cdot 11^3 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23}{67 \cdot 1999 \cdot 2^{30} 3^{15} 5^7 7^5 11}, \\
\frac{b_7}{c_7} &= \frac{2^{56} \cdot 3^{28} \cdot 5^{13} \cdot 7^8 \cdot 11^4 \cdot 13^3 \cdot 17^2 \cdot 19^2 \cdot 23}{43 \cdot 46663 \cdot 2^{42} 3^{21} 5^9 7^7 11^3 13}, \quad \frac{b_8}{c_8} = \frac{2^{72} \cdot 3^{34} \cdot 5^{16} \cdot 7^{11} \cdot 11^6 \cdot 13^4 \cdot 17^3 \cdot 19^2 \cdot 23 \cdot 29 \cdot 31}{46743947 \cdot 2^{56} 3^{28} 5^{12} 7^9 11^5 13^3}, \\
\frac{b_9}{c_9} &= \frac{2^{90} \cdot 3^{42} \cdot 5^{21} \cdot 7^{14} \cdot 11^8 \cdot 13^6 \cdot 17^3 \cdot 19^3 \cdot 23^2 \cdot 29 \cdot 31}{19583 \cdot 16249 \cdot 2^{72} 3^{36} 5^{16} 7^{11} 11^7 13^5 17}, \quad \frac{b_{10}}{c_{10}} = \frac{2^{110} \cdot 3^{55} \cdot 5^{25} \cdot 7^{17} \cdot 11^{10} \cdot 13^8 \cdot 17^5 \cdot 19^4 \cdot 23^3 \cdot 29 \cdot 31 \cdot 37}{3156627824489 \cdot 2^{90} 3^{44} 5^{20} 7^{13} 11^9 13^7 17^3 19}, \\
\frac{b_{11}}{c_{11}} &= \frac{2^{132} \cdot 3^{63} \cdot 5^{31} \cdot 7^{18} \cdot 11^{12} \cdot 13^{10} \cdot 17^5 \cdot 19^5 \cdot 23^4 \cdot 29^2 \cdot 31^2 \cdot 37 \cdot 41 \cdot 43}{59 \cdot 11332613 \cdot 33391 \cdot 2^{110} 3^{53} 5^{24} 7^{16} 11^{11} 13^9 17^5 19^3}, \\
\frac{b_{12}}{c_{12}} &= \frac{2^{156} \cdot 3^{75} \cdot 5^{37} \cdot 7^{23} \cdot 11^{15} \cdot 13^{12} \cdot 17^8 \cdot 19^7 \cdot 23^4 \cdot 29^3 \cdot 31^2 \cdot 41 \cdot 43 \cdot 47}{241 \cdot 251799899121593 \cdot 2^{132} 3^{63} 5^{28} 7^{20} 11^{13} 13^{11} 17^7 19^5 23}, \\
\frac{b_{13}}{c_{13}} &= \frac{2^{182} \cdot 3^{90} \cdot 5^{42} \cdot 7^{28} \cdot 11^{17} \cdot 13^{14} \cdot 17^{10} \cdot 19^8 \cdot 23^5 \cdot 29^3 \cdot 31^3 \cdot 37^2 \cdot 41 \cdot 43 \cdot 47}{285533 \cdot 37408704134429 \cdot 2^{156} 3^{73} 5^{34} 7^{24} 11^{15} 13^{13} 17^9 19^7 23^3}, \\
\frac{b_{14}}{c_{14}} &= \frac{2^{210} \cdot 3^{100} \cdot 5^{50} \cdot 7^{31} \cdot 11^{20} \cdot 13^{17} \cdot 17^{12} \cdot 19^{10} \cdot 23^7 \cdot 29^4 \cdot 31^4 \cdot 37^2 \cdot 41 \cdot 43 \cdot 47 \cdot 53}{197 \cdot 1462253323 \cdot 6616773091 \cdot 2^{182} 3^{86} 5^{42} 7^{28} 11^{17} 13^{15} 17^{11} 19^9 23^5}, \\
\frac{b_{15}}{c_{15}} &= \frac{2^{240} \cdot 3^{117} \cdot 5^{57} \cdot 7^{37} \cdot 11^{22} \cdot 13^{19} \cdot 17^{14} \cdot 19^{11} \cdot 23^9 \cdot 29^5 \cdot 31^5 \cdot 37^3 \cdot 41^2 \cdot 43^2 \cdot 47 \cdot 53 \cdot 59}{16255375825174687265195458 37 \cdot 2^{210} 3^{102} 5^{50} 7^{32} 11^{19} 13^{17} 17^{13} 19^{11} 23^7 29}.
\end{aligned}
$$

Table 4. The values of $b(k)/c(k)$ for $k = 1, 2, ..., 15$.

Using the formula (2.24), the values of $b(k)/c(k)$ in Table 4, and the values of $A(k)$ in Table 2, we have the following new lower bounds for $\Lambda(k)$ for $k = 1, 2, ..., 15$:

| | | | | |
|---|---|---|---|---|
| $\Lambda(1)$ | $\Lambda(2)$ | $\Lambda(3)$ | $\Lambda(4)$ | $\Lambda(5)$ |
| 1.732 1 | 2.2635 | 2.7080 | 3.1257 | 3.5177 |
| $\Lambda(6)$ | $\Lambda(7)$ | $\Lambda(8)$ | $\Lambda(9)$ | $\Lambda(10)$ |
| 3.8813 | 4.2150 | 4.5196 | 4.7985 | 5.0560 |
| $\Lambda(11)$ | $\Lambda(12)$ | $\Lambda(13)$ | $\Lambda(14)$ | $\Lambda(15)$ |
| 5.2962 | 5.5225 | 5.7373 | 5.9424 | 6.1392 |

Table 4. The lower bounds for $\Lambda(k)$ for $k = 1, 2, ..., 15$.

From this table we have the following theorem.

**Theorem 2.5.** *On the hypothesis that the Riemann hypothesis is true, (1.9) and (1.11) are correctly predicted, we have*

$$\Lambda \geq 6.1392 \tag{2.26}$$

**Remark 1.** *Using the explicit formulae for the $b(k)$ and $c(k)$ (which would via (2.24) help to decide whether the conjecture $\Lambda = \infty$ is true subject to the Riemann hypothesis),*

*we have the following formula*

$$\begin{aligned}\Lambda(k) \quad &\geq \quad \frac{1}{\pi A(k)} \left( \prod_{j=0}^{k-1} \frac{j!}{(j+k)!} \right)^{\frac{1}{2k}} \\ &\times (-1)^{\frac{k(k+1)}{2}} \sum_{m \in P_O^{k+1}(2k)}^{k} \binom{2k}{m} \left( \frac{-1}{2} \right)^{m_0} \left( \prod_{i=1}^{k} \frac{1}{(2k-i+m_i)!} \right) M_{i,j},\end{aligned}$$

*where* $A(k)$ *is defined as in (2.13),* $M_{i,j}$ *as defined in (1.12) and* $P_O^{k+1}(2k)$ *denotes the set of partitions* $m = (m_0, ..., m_k)$ *of* $2k$ *into nonnegative parts.*

**Remark 2.** *The lower bound in (2.26) means that consecutive nontrivial zeros often differ by at least* 6.1392 *times the average spacing. This value improves the value of* $\Lambda \geq 4.71474396$ *that has been obtained by Hall.*

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